Math 122 **Name:** ____________________

Spring 2024

Exam 2 practice

3/14/2024

Time Limit: 75 Minutes Signature: ____________________

This exam has 8 questions, for a total of 80 points and 0 bonus points. Unless otherwise specified, there is no form of technology allowed. Further, final solutions must be written in the prescribed boxes, and all work must be shown. There is paper provided in the front of the class for scratch work. Any numerical values given for a final answer must be precise. *The actual format of the exam is not a direct reflection of this practice.*

Grade Table (for teacher use only)

Question	Points	Bonus Points	Score
1	10	0	
2	10	0	
3	10	0	
4	10	0	
5	10	0	
6	10	0	
7	10	0	
8	10	0	
Total:	80	0	

1. (10 points) Show the following using the definition of a derivative as a limit (i.e. §2 theory section): If $f(x) = x^5$, then $f'(x) = 5x^4$. (§3 on theory)

Case: $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$[(x+h)^3 = (x+h)^2(x+h)$
$= (x^2+2xh+h^2)(x+h)$
$= x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3]$

$$= \lim_{h\to 0} (\cancel{x^3} + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3 - \cancel{x^3})\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x^2 + xh + x^2 + 2xh + h^2)}{\cancel{h}} = \lim_{h\to 0} 2x^2 + x^2 = 3x^2.$$

2. (10 points) Compute the derivative of $f(t) = \frac{e^t}{1+e^t}$. $(= e^t \cdot (1+e^t)^{-1})$

$f(t) = \frac{e^t}{1+e^t} = \frac{g(t)}{h(t)}$, so $g(t) = e^t$, $h(t) = 1+e^t$

$$\boxed{f'(t) = \left(\frac{g}{h}\right)' = \frac{g' \cdot h - gh'}{h^2}} = \frac{e^t(1+e^t) - e^t \cdot e^t}{(1+e^t)^2}$$

$$= \frac{e^t + \cancel{e^{2t}} - \cancel{e^{2t}}}{(1+e^t)^2} = \frac{e^t}{(1+e^t)^2}$$

3. (10 points) Compute the derivative of $g(x) = x^2 \ln(x)$.

$g(x) = x^2 \ln(x) = f(x)\,h(x)$ w/ $f(x) = x^2$, $h(x) = \ln(x)$

$\boxed{g'(x) = f'h + fh'} = 2x\ln(x) + x^2 \cdot \frac{1}{x}$

$= 2x\ln(x) + x$

$= x\,(1 + 2\ln(x))$

$\lceil a + ab = a(1+b) \rfloor$

4. (10 points) Compute the derivative of $h(x) = \sqrt{e^x + 1}$.

$h(x) = f(g(x))$ w/ $f(x) = \sqrt{x}$, $g(x) = e^x + 1$

$\boxed{h'(x) = f'(g(x)) \cdot g'(x)}$ $\left(\frac{d}{dx}h = \frac{d}{dx} f\Big|_{x=g(x)} \cdot \frac{d}{dx}g \right)$

$f'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}\,x^{-\frac{1}{2}}$

$f'(g(x)) = f'(e^x + 1) = \frac{1}{2}\,(e^x + 1)^{-\frac{1}{2}}$

$g'(x) = \frac{d}{dx}(e^x + 1) = e^x$

$h'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2}(e^x + 1)^{-\frac{1}{2}} \cdot e^x$

$= \frac{1}{2} \frac{e^x}{\sqrt{e^x + 1}}$

5. (10 points) Compute the derivative of $i(x) = 2^x + 2 \cdot 3^x$.

$\boxed{\left(\frac{d}{dx}(a^x) = \ln(a) \cdot a^x\right)}$

$$\frac{d}{dx} i(x) = \frac{d}{dx}\left(2^x + 2 \cdot 3^x\right) = \frac{d}{dx}(2^x) + \frac{d}{dx}(2 \cdot 3^x)$$

$$= \ln(2)\, 2^x + 2 \frac{d}{dx}(3^x)$$

$\boxed{\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x)}$

$$= \ln(2)\, 2^x + 2 \ln(3)\, 3^x$$

6. (10 points) Compute the derivative of $j(x) = x^2 - 2\ln(x)$.

$$\frac{d}{dx} j(x) = \frac{d}{dx}\left(x^2 - 2\ln(x)\right) =$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(-2\ln(x)) =$$

$$2x - 2\frac{d}{dx}(\ln(x)) =$$

$\boxed{\frac{d}{dx} \ln(x) = \frac{1}{x}}$

$$2x - 2\frac{1}{x} =$$

$$2\left(x - \frac{1}{x}\right)$$

7. (10 points) Compute the derivative of $k(x) = \sqrt{x}(x+1)$

$$\frac{d}{dx} k(x) = \frac{d}{dx}\left(\sqrt{x}\cdot(x+1)\right)$$

* Try to write all in one sweep *

$$= \frac{d}{dx}(\sqrt{x})\cdot(x+1) + \sqrt{x}\,\frac{d}{dx}(x+1)$$

$$= \frac{1}{2}x^{-\frac{1}{2}}(x+1) + \sqrt{x}\cdot 1$$

$$= \frac{1}{2}\,\frac{x+1}{\sqrt{x}} + \sqrt{x}$$

8. (10 points) Compute the second derivative of $f(x) = t^4 - 3t^2 + 5t$

$$\frac{d^2}{dx^2} f(x) = \frac{d}{dx}\left(\frac{d}{dx} f\right) = \frac{d}{dx}\left(\boxed{\frac{d}{dx}\left(x^4 - 3x^2 + 5x\right)}\right)$$

$$= \frac{d}{dx}\left(\frac{d}{dx}(x^4) + \frac{d}{dx}(-3x^2) + \frac{d}{dx}(5x)\right)$$

$$= \frac{d}{dx}\left(4x^3 - 6x + 5\right)$$

$$= \frac{d}{dx}(4x^3) + \frac{d}{dx}(-6x) + \cancel{\frac{d}{dx}(5)}$$

$$= 12x^2 - 6$$

$$= 6(2x^2 - 1)$$

Review of derivatives

$f(x), g(x)$ are functions

$c, n \in (-\infty, \infty)$, $a \in (0, \infty) \setminus \{1\}$

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{d}{dx}(a^x) = \ln(a) \cdot a^x$$

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x)$$

$$\frac{d}{dx}(f \pm g) = \frac{d}{dx} f \pm \frac{d}{dx} g$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

$$\frac{d}{dx}(f(g(x))) = (f \circ g)' = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}(f \cdot g) = \frac{d}{dx}(f) \cdot g + f \cdot \frac{d}{dx}(g)$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}(f) \cdot g - f \frac{d}{dx}(g)}{g^2} \qquad (g \neq 0)$$

Extra problems : (§3 focus on practice)

5a: $$\frac{d}{dz}\left(\frac{z^2+1}{\sqrt{z}}\right) = \frac{\boxed{\frac{d}{dz}(z^2+1)} \cdot \sqrt{z} - (z^2+1) \cdot \frac{d}{dz}(\sqrt{z})}{(\sqrt{z})^2}$$

$$= \frac{2z\sqrt{z} - (z^2+1)\frac{1}{2}z^{-\frac{1}{2}}}{z}$$

$$= \frac{2\cancel{z}\sqrt{z}}{\cancel{z}} - \frac{(z^2+1)}{2z\underbrace{\sqrt{z}}}$$

$$= 2\sqrt{z} - \frac{z^2+1}{2z^{3/2}}$$

<u>41</u>: $\frac{d}{dw}\left((w^4-2w)^5\right) =$

$\boxed{5(w^4-2w)^4}\ \boxed{\frac{d}{dw}(w^4-2w)} =$

$5(w^4-2w)^4 \cdot \left(\frac{d}{dw}(w^4) - \frac{d}{dw}(2w)\right) =$

$5(w^4-2w)^4(\underbrace{4w^3-2}) =$

$10(w^4-2w)^4(2w^3-1)$

$\left[\ \frac{d}{dx}(f \circ g) = (f \circ g)' = \boxed{f'(g(x))} \cdot \boxed{g'(x)}\ \right]$

<u>22</u>: $\frac{d}{dt}\left((e^t+4)^3\right) =$

$3(e^t+4)^2 \cdot \frac{d}{dt}(e^t+4) =$

$3(e^t+4)^2 \cdot e^t$